

Fuzzy h -ideals of a Γ -hemiring and its operator hemirings

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Abstract

Various correspondence between fuzzy h -ideals of a Γ -hemiring and fuzzy h -ideals of its operator hemirings are established and some of their characterization are given using lattice structure and cartesian product.

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1 Introduction

Hemirings [3] which provide a common generalization of rings and distributive lattices arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, mathematical modelling and parallel computation systems etc.(for example, see [3], [4]). Hemirings have also been proved to be an important algebraic tool in theoretical computer science, see for instance [4], for some detail and example. Ideals of

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semiring(hemiring) play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To solve this problem, Henriksen [5], defined a more restricted class of ideals, which are called k -ideals. A still more restricted class of ideals in hemirings are given by Iizuka [6], which are called h -ideals. LaTorre [8], investigated h -ideals and k -ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring. The theory of Γ -semiring was introduced by Rao [10] as a generalization of semiring. The notion of Γ -semiring theory has been enriched by the introduction of operator semirings of a Γ -semiring by Dutta and Sardar[2]. To make operator semirings effective in the study of Γ -semirings Dutta et. al. [2] established a correspondence between the ideals of a Γ -semiring S and the ideals of the operator semirings of S .

The concept of fuzzy set was introduced by Zadeh[13] and has been applied to many branches of mathematics. The theory of fuzzy h -ideals in hemiring was introduced and studied by Jun et. al. [7], Zhan et. al. [14]. As a continuation of this Sardar et al[12] studied those properties in Γ -hemiring in terms of fuzzy h -ideals. Recently Ma et. al. [9] investigated some properties of fuzzy h -ideals in Γ -hemirings. In this paper we establish various correspondence between the fuzzy h -ideals of a Γ -hemiring S and the fuzzy h -ideals of the operator hemirings of S .

2 Preliminaries

We recall the following definition from [3].

A *hemiring* (respectively, *semiring*) is a non-empty set S on which operations addition and multiplication have been defined such that $(S, +)$ is a commutative monoid with identity 0, (S, \cdot) is a semigroup (respectively, monoid with identity 1_S), multiplication distributes over addition from either side, $1_S \neq 0$ and $0s = 0 = s0$ for all $s \in S$.

Let S and Γ be two additive commutative semigroups with zero. According to [12], S is called a Γ -*hemiring* if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ by $(a, \alpha, b) \mapsto a\alpha b$ satisfying the following conditions:

- (1) $(a + b)\alpha c = a\alpha c + b\alpha c$,
- (2) $a\alpha(b + c) = a\alpha b + a\alpha c$,
- (3) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
- (4) $a\alpha(b\beta c) = (a\alpha b)\beta c$,
- (5) $0_S\alpha a = 0_S = a\alpha 0_S$,

$$(6) \quad a0_\Gamma b = 0_S = b0_\Gamma a,$$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

For simplification we write 0 instead of 0_S and 0_Γ .

Let S be the set of all $m \times n$ matrices over \mathbf{Z}_0^- (the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over \mathbf{Z}_0^- . Then S forms a Γ -hemiring with usual addition and multiplication of matrices.

Now, we recall the following definitions from [2].

Let S be a Γ -hemiring and F be the free additive commutative semigroup generated by $S \times \Gamma$. We define a relation ρ on F as follows:

$$\sum_{i=1}^m (x_i, \alpha_i) \rho \sum_{j=1}^n (y_j, \beta_j) \text{ if and only if } \sum_{i=1}^m x_i \alpha_i a = \sum_{j=1}^n y_j \beta_j a,$$

for all $a \in S$ ($m, n \in \mathbf{Z}^+$). Then ρ is a congruence relation on F . We denote the congruence class containing $\sum_{i=1}^m (x_i, \alpha_i)$ by $\sum_{i=1}^m [x_i, \alpha_i]$. Then F/ρ is an additive commutative semigroup. Now, F/ρ forms a hemiring with the multiplication defined by

$$\left(\sum_{i=1}^m [x_i, \alpha_i] \right) \left(\sum_{j=1}^n [y_j, \beta_j] \right) = \sum_{i,j} [x_i \alpha_i y_j, \beta_j].$$

We denote this hemiring by L and call it the *left operator hemiring* of the Γ -hemiring S . Dually we define the *right operator hemiring* R of the Γ -hemiring S . Let S be a Γ -hemiring and L be the left operator hemiring and R be the right one. If there exists an element $\sum_{i=1}^m [e_i, \delta_i] \in L$ (resp. $\sum_{j=1}^n [\gamma_j, f_j] \in R$) such

that $\sum_{i=1}^m e_i \delta_i a = a$ (respectively, $\sum_{j=1}^n a \gamma_j f_j = a$) for all $a \in S$, then S is said to

have the *left unity* $\sum_{i=1}^m [e_i, \delta_i]$ (respectively, the *right unity* $\sum_{j=1}^n [\gamma_j, f_j]$).

Throughout this paper unless otherwise mentioned for different elements of L (respectively, R) we take the same index say ' i ' whose range is finite that is from 1 to n , for some positive integer n .

Let S be a Γ -hemiring, L be the left operator hemiring and R be the right one. If there exists an element $[e, \delta] \in L$ (respectively, $[\gamma, f] \in R$) such that $e \delta a = a$ (respectively, $a \gamma f = a$) for all $a \in S$, then S is said to have the *strong left unity* $[e, \delta]$ (respectively, *strong right unity* $[\gamma, f]$) [10].

Let S be a Γ -hemiring, L be the left operator hemiring and R be the right one. Let $P \subseteq L$ ($\subseteq R$). According to [2], we define $P^+ = \{a \in S : [a, \Gamma] \subseteq P\}$

(respectively, $P^* = \{a \in S : [\Gamma, a] \subseteq P\}$) and for $Q \subseteq S$,

$$Q^{+'} = \left\{ \sum_{i=1}^m [x_i, \alpha_i] \in L : \left(\sum_{i=1}^m ([x_i, \alpha_i]) \right) S \subseteq Q \right\},$$

where $\left(\sum_{i=1}^m [x_i, \alpha_i] \right) S$ denotes the set of all finite sums $\sum_{i,k} x_i \alpha_i s_k$, $s_k \in S$ and

$$Q^{*'} = \left\{ \sum_{i=1}^m [\alpha_i, x_i] \in R : S \left(\sum_{i=1}^m ([\alpha_i, x_i]) \right) \subseteq Q \right\},$$

where $S \left(\sum_{i=1}^m [x_i, \alpha_i] \right)$ denotes the set of all finite sums $\sum_{i,k} s_k \alpha_i x_i$, $s_k \in S$.

A *fuzzy subset* μ of a non-empty set S is a function $\mu : S \rightarrow [0, 1]$.

Let μ be a non-empty fuzzy subset of a Γ -hemiring S (i.e., $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a *fuzzy left ideal* (respectively, *fuzzy right ideal*) of S [12] if

- (1) $\mu(x + y) \geq \min[\mu(x), \mu(y)]$,
- (2) $\mu(x\gamma y) \geq \mu(y)$ (respectively, $\mu(x\gamma y) \geq \mu(x)$),

for all $x, y \in S$ and $\gamma \in \Gamma$. A *fuzzy ideal* of a Γ -hemiring S is a non-empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S . Note that if μ is a fuzzy left or right ideal of a Γ -hemiring S , then $\mu(0) \geq \mu(x)$ for all $x \in S$.

A left ideal A of a Γ -hemiring S is called a *left h-ideal* if for any $x, z \in S$ and $a, b \in A$,

$$x + a + z = b + z \implies x \in A.$$

A *right h-ideal* is defined analogously. A fuzzy left ideal μ of a Γ -hemiring S is called a *fuzzy left h-ideal* if for all $a, b, x, z \in S$,

$$x + a + z = b + z \implies \mu(x) \geq \min\{\mu(a), \mu(b)\}.$$

A *fuzzy right h-ideal* is defined similarly. By a *fuzzy h-ideal* μ , we mean that μ is both fuzzy left and fuzzy right *h-ideal*.

For example, let S be the additive commutative semigroup of all non-positive integers and Γ be the additive commutative semigroup of all non-positive even integers. Then S is a Γ -hemiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of S , defined as follows

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.7 & \text{if } x \text{ is even} \\ 0.1 & \text{if } x \text{ is odd} \end{cases}$$

The fuzzy subset μ of S is both a fuzzy ideal and a fuzzy h -ideal of S .

Let S be a Γ -hemiring and μ_1, μ_2 be two fuzzy subsets of S . Then the sum $\mu_1 \oplus \mu_2$ is defined as follows:

$$(\mu_1 \oplus \mu_2)(x) = \begin{cases} \sup_{x=u+v} \{\min\{\mu_1(u), \mu_2(v)\} : u, v \in S\} \\ 0 \text{ if } x \text{ can not be expressed as above.} \end{cases}$$

Let μ and θ be two fuzzy subsets of a Γ -hemiring S . We define *generalized h -product* of μ and θ by

$$\mu \circ_h \theta(x) = \begin{cases} \sup \left\{ \min_i \{\mu(a_i), \mu(c_i), \theta(b_i), \theta(d_i)\} : x + \sum_{i=1}^n a_i \gamma_i b_i + z = \sum_{i=1}^n c_i \delta_i d_i + z \right\} \\ 0 \text{ if } x \text{ can not be expressed as above,} \end{cases}$$

where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \dots, n$.

Ma et. al. [9] also defined simple h -product by

$$\mu \Gamma_h \theta(x) = \begin{cases} \sup \{\min\{\mu(a), \mu(c), \theta(b), \theta(d)\} : x + a\gamma b + z = c\delta d + z\} \\ 0 \text{ if } x \text{ can not be expressed as above,} \end{cases}$$

where $x, z, a, c, d \in S$ and $\gamma, \delta \in \Gamma$.

We now recall following two definitions from [9]

A fuzzy left(right) h -ideal ζ of a Γ -hemiring S is said to be *prime* if ζ is a non-constant function and for any two fuzzy left(right) h -ideals μ and ν of S , $\mu \Gamma_h \nu \subseteq \zeta$ implies $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Similarly, we can define *semiprime* fuzzy h -ideal.

A fuzzy subset μ of a Γ -hemiring S is called *fuzzy h -bi-ideal* if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,
2. $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$,
3. $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$,
4. $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

A fuzzy subset μ of a Γ -hemiring S is called *fuzzy h -quasi-ideal* if for all $x, y, z, a, b \in S$ we have

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,
2. $(\mu \circ_h \chi_S) \cap (\chi_S \circ_h \mu) \subseteq \mu$,
3. $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

For more preliminaries of semirings (hemirings) and Γ -semirings we refer to [3] and [2], respectively. Also, for more results on fuzzy h -ideals in Γ -hemirings we refer to [12]. Throughout this paper unless otherwise mentioned S denotes a Γ -hemiring with left unity and right unity and $\text{FLh-I}(S)$, $\text{FRh-I}(S)$ and $\text{Fh-I}(S)$ denote respectively the set of all fuzzy left h -ideals, the set of all fuzzy right h -ideals and the set of all fuzzy h -ideals of the Γ -hemiring S . Similar is the meaning of $\text{FLh-I}(L)$, $\text{FLh-I}(R)$, $\text{FRh-I}(L)$, $\text{FRh-I}(R)$, $\text{Fh-I}(L)$, $\text{Fh-I}(R)$, where L and R are respectively the left operator and right operator hemirings of the Γ -hemiring S . Also, in this section we assume that $\mu(0) = 1$ for a fuzzy left h -ideal (respectively, fuzzy right h -ideal, fuzzy h -ideal) μ of a Γ -hemiring S . Similarly, we assume that $\mu(0_L) = 1$ (respectively, $\mu(0_R) = 1$) for a fuzzy left h -ideal (respectively, fuzzy right h -ideal, fuzzy h -ideal) μ of the left operator hemiring (respectively, right operator hemiring R) of a Γ -hemiring S .

3 Correspondence of Fuzzy h -ideals

Throughout this section S denotes a Γ -hemiring, R denotes the right operator hemiring and L denotes the left operator hemiring of the Γ -hemiring S . Now, we recall the following definitions from [11].

Definition 3.1. Let μ be a fuzzy subset of L . We define a fuzzy subset μ^+ of S by

$$\mu^+(x) = \inf_{\gamma \in \Gamma} \{ \mu([x, \gamma]) \},$$

where $x \in S$. If σ is a fuzzy subset of S , we define a fuzzy subset $\sigma^{+'}$ of L by

$$\sigma^{+'} \left(\sum_i [x_i, \alpha_i] \right) = \inf_{s \in S} \left\{ \sigma \left(\sum_i x_i \alpha_i s \right) \right\},$$

where $\sum_i [x_i, \alpha_i] \in L$.

Definition 3.2. Let δ be a fuzzy subset of R . We define a fuzzy subset δ^* of S by

$$\delta^*(x) = \inf_{\gamma \in \Gamma} \{ \delta([\gamma, x]) \},$$

where $x \in S$. If η is a fuzzy subset of S , we define a fuzzy subset $\eta^{*'}$ of R by

$$\eta^{*'} \left(\sum_i [\alpha_i, x_i] \right) = \inf_{s \in S} \left\{ \eta \left(\sum_i s \alpha_i x_i \right) \right\},$$

where $\sum_i [\alpha_i, x_i] \in R$.

Lemma 3.3. If $\{\mu_i : i \in I\}$ is a collection of fuzzy subsets of L , then

$$\bigcap_{i \in I} \mu_i^+ = \left(\bigcap_{i \in I} \mu_i \right)^+.$$

Proof. It is straightforward. \square

Proposition 3.4. If $\mu \in \text{Fh-I}(L)$, then $\mu^+ \in \text{Fh-I}(S)$.

Proof. Let $\mu \in \text{Fh-I}(L)$. Then $\mu(0_L) = 1$. Since for all $\gamma \in \Gamma$, $[0_S, \gamma]$ is the zero element of L , we have $\mu^+(0_S) = \inf_{\gamma \in \Gamma} \{\mu([0_S, \gamma])\} = 1$. So μ^+ is non empty and $\mu^+(0_S) = 1$. Let $x, y \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \mu^+(x + y) &= \inf_{\gamma \in \Gamma} \{\mu([x + y, \gamma])\} \\ &= \inf_{\gamma \in \Gamma} \{\mu([x, \gamma] + [y, \gamma])\} \\ &\geq \inf_{\gamma \in \Gamma} \{\min\{\mu([x, \gamma]), \mu([y, \gamma])\}\} \\ &= \min \left\{ \inf_{\gamma \in \Gamma} \{\mu([x, \gamma])\}, \inf_{\gamma \in \Gamma} \{\mu([y, \gamma])\} \right\} \\ &= \min\{\mu^+(x), \mu^+(y)\}. \end{aligned}$$

Therefore, $\mu^+(x + y) \geq \min\{\mu^+(x), \mu^+(y)\}$. Also, we have

$$\mu^+(x\alpha y) = \inf_{\gamma \in \Gamma} \{\mu([x\alpha y, \gamma])\} \geq \inf_{\gamma \in \Gamma} \{\mu([x, \alpha][y, \gamma])\} \geq \inf_{\gamma \in \Gamma} \{\mu[y, \gamma]\} = \mu^+(y)$$

and

$$\mu^+(x\alpha y) = \inf_{\gamma \in \Gamma} \{\mu([x\alpha y, \gamma])\} = \inf_{\gamma \in \Gamma} \{\mu([x, \alpha][y, \gamma])\} \geq \mu([x, \alpha]) \geq \inf_{\delta \in \Gamma} \{\mu([x, \delta])\} = \mu^+(x).$$

Hence, μ^+ is a fuzzy ideal of S . Now, suppose that $x + a + z = b + z$, for $x, a, b, z \in S$. Then

$$\begin{aligned} \mu^+(x) &= \inf_{\gamma \in \Gamma} \{\mu([x, \gamma])\} \\ &\geq \inf_{\gamma \in \Gamma} \{\min\{\mu([a, \gamma]), \mu([b, \gamma])\}\} \\ &= \min \left\{ \inf_{\gamma \in \Gamma} \{\mu([a, \gamma])\}, \inf_{\gamma \in \Gamma} \{\mu([b, \gamma])\} \right\} \\ &= \min\{\mu^+(a), \mu^+(b)\}. \end{aligned}$$

Therefore, μ^+ is a fuzzy h -ideal of S . \square

Proposition 3.5. If $\sigma \in \text{Fh-I}(S)$ (respectively, $\text{FRh-I}(S)$, $\text{FLh-I}(S)$), then $\sigma^{+'} \in \text{Fh-I}(L)$ (respectively, $\text{FRh-I}(L)$, $\text{FLh-I}(L)$).

Proof. Let $\sigma \in \text{Fh-I}(S)$. Then $\sigma(0_S) = 1$. Now, we have

$$\sigma^{+'}([0_S, \gamma]) = \inf_{s \in S} \{\sigma(0_S \gamma s)\} = \inf_{s \in S} \{\sigma(0_S)\} = 1,$$

for all $\gamma \in \Gamma$. Therefore, $\sigma^{+'}$ is non-empty and $\sigma^{+'}(0_L) = 1$ as $[0_S, \gamma]$ is the zero element of L . Let $\sum_i [x_i, \alpha_i], \sum_j [y_j, \beta_j] \in L$. Then

$$\begin{aligned} & \sigma^{+'} \left(\sum_i [x_i, \alpha_i] + \sum_j [y_j, \beta_j] \right) \\ &= \inf_{s \in S} \left\{ \sigma \left(\sum_i x_i \alpha_i s + \sum_j y_j \beta_j s \right) \right\} \\ &\geq \inf_{s \in S} \left\{ \min \left\{ \sigma \left(\sum_i x_i \alpha_i s \right), \sigma \left(\sum_j y_j \beta_j s \right) \right\} \right\} \\ &= \min \left\{ \inf_{s \in S} \left\{ \sigma \left(\sum_i x_i \alpha_i s \right) \right\}, \inf_{s \in S} \left\{ \sigma \left(\sum_j y_j \beta_j s \right) \right\} \right\} \\ &= \min \left\{ \sigma^{+'} \left(\sum_i [x_i, \alpha_i] \right), \sigma^{+'} \left(\sum_j [y_j, \beta_j] \right) \right\}. \end{aligned}$$

Also, we have

$$\begin{aligned} & \sigma^{+'} \left(\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j] \right) \\ &= \sigma^{+'} \left(\sum_{i,j} [x_i \alpha_i y_j, \beta_j] \right) \\ &= \inf_{s \in S} \left\{ \sigma \left(\sum_{i,j} x_i \alpha_i y_j \beta_j s \right) \right\} \\ &\geq \inf_{s \in S} \left\{ \min \left\{ \sigma \left(\sum_i x_i \alpha_i y_1 \right), \sigma \left(\sum_i x_i \alpha_i y_2 \right), \sigma \left(\sum_i x_i \alpha_i y_3 \right), \dots \right\} \right\} \\ &\geq \min \left\{ \sigma \left(\sum_i x_i \alpha_i y_1 \right), \sigma \left(\sum_i x_i \alpha_i y_2 \right), \sigma \left(\sum_i x_i \alpha_i y_3 \right), \dots \right\} \\ &\geq \inf_{s \in S} \left\{ \sigma \left(\sum_i (x_i \alpha_i s) \right) \right\} \\ &= \sigma^{+'} \left(\sum_i [x_i, \alpha_i] \right). \end{aligned}$$

Similarly, we can show that

$$\sigma^{+'} \left(\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j] \right) \geq \sigma^{+'} \left(\sum_j [y_j, \beta_j] \right).$$

Thus, $\sigma^{+'}$ is a fuzzy ideal of L . Now, suppose that

$$\sum_i [x_i, e_i] + \sum_i [a_i, \alpha_i] + \sum_i [z_i, \delta_i] = \sum_i [b_i, \beta_i] + \sum_i [z_i, \delta_i],$$

where $\sum_i [x_i, e_i], \sum_i [a_i, \alpha_i], \sum_i [z_i, \delta_i], \sum_i [b_i, \beta_i] \in L$. Then

$$\begin{aligned} \sigma^{+'} \left(\sum_i [x_i, e_i] \right) &= \inf_{s \in S} \left\{ \sigma \left(\sum_i x_i e_i s \right) \right\} \\ &\geq \inf_{s \in S} \left\{ \min \left\{ \sigma \left(\sum_i a_i \alpha_i s \right), \sigma \left(\sum_j b_j \beta_j s \right) \right\} \right\} \\ &= \min \left\{ \inf_{s \in S} \left\{ \sigma \left(\sum_i a_i \alpha_i s \right) \right\}, \inf_{s \in S} \left\{ \sigma \left(\sum_j b_j \beta_j s \right) \right\} \right\} \\ &= \min \left\{ \sigma^{+'} \left(\sum_i [a_i, \alpha_i] \right), \sigma^{+'} \left(\sum_j [b_j, \beta_j] \right) \right\}. \end{aligned}$$

Therefore, $\sigma^{+'}$ is a fuzzy h -ideal of L . □

Similarly, we can prove the following propositions.

Proposition 3.6. If $\delta \in \text{Fh-I}(R)$ (respectively, $\text{FRh-I}(R)$, $\text{FLh-I}(R)$), then $\delta^* \in \text{Fh-I}(S)$ (respectively, $\text{FRh-I}(S)$, $\text{FLh-I}(S)$).

Proposition 3.7. If $\eta \in \text{Fh-I}(S)$ (respectively, $\text{FRh-I}(S)$, $\text{FLh-I}(S)$), then $\eta^{*'} \in \text{Fh-I}(R)$ (respectively, $\text{FRh-I}(R)$, $\text{FLh-I}(R)$).

Theorem 3.8. The lattice of all fuzzy h -ideals of S and the lattice of all fuzzy h -ideals of L are isomorphic via the mapping $\sigma \mapsto \sigma^{+'}$, where $\sigma \in \text{Fh-I}(S)$ and $\sigma^{+'} \in \text{Fh-I}(L)$.

Proof. First, we show that $(\sigma^{+'})^+ = \sigma$, where $\sigma \in \text{Fh-I}(S)$. Let $x \in S$. Then

$$((\sigma^{+'})^+)(x) = \inf_{\gamma \in \Gamma} \{ \sigma^{+'}([x, \gamma]) \} = \inf_{\gamma \in \Gamma} \left\{ \inf_{s \in S} \{ \sigma(x\gamma s) \} \right\} \geq \inf_{\gamma \in \Gamma} \left\{ \inf_{s \in S} \{ \sigma(x) \} \right\} = \sigma(x).$$

So $\sigma \subseteq (\sigma^{+'})^+$. Now, let $\sum_i [\gamma_i, f_i]$ be the right unity of S . Then $\sum_i x\gamma_i f_i = x$ for all $x \in S$. We have

$$\sigma(x) = \sigma \left(\sum_i x\gamma_i f_i \right) \geq \min \{ \sigma(x\gamma_1 f_1), \sigma(x\gamma_2 f_2), \dots \} \geq \inf_{\gamma \in \Gamma} \left\{ \inf_{s \in S} \{ \sigma(x\gamma s) \} \right\} = (\sigma^{+'})^+(x).$$

Therefore, $(\sigma^{+'})^+ \subseteq \sigma$ and so $(\sigma^{+'})^+ = \sigma$. Now, let $\mu \in \text{Fh-I}(L)$. Then

$$\begin{aligned}
((\mu^+)^{+'}) \left(\sum_i [x_i, \alpha_i] \right) &= \inf_{s \in S} \left\{ \mu^+ \left(\sum_i x_i \alpha_i s \right) \right\} \\
&= \inf_{s \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left(\sum_i [x_i \alpha_i s, \gamma] \right) \right\} \right\} \\
&= \inf_{s \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left(\sum_i [x_i, \alpha_i][s, \gamma] \right) \right\} \right\} \\
&\geq \inf_{s \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left(\sum_i [x_i, \alpha_i] \right) \right\} \right\} \\
&= \mu \left(\sum_i [x_i, \alpha_i] \right).
\end{aligned}$$

So $\mu \subseteq (\mu^+)^{+}$. Let $\sum_i [e_i, \delta_i]$ be the left unity of S . Then

$$\begin{aligned}
\mu \left(\sum_j [x_j, \alpha_j] \right) &= \mu \left(\sum_j [x_j, \alpha_j] \sum_i [e_i, \delta_i] \right) \\
&\geq \min \left\{ \mu \left(\sum_j [x_j, \alpha_j][e_1, \delta_1] \right), \mu \left(\sum_j [x_j, \alpha_j][e_2, \delta_2] \right), \dots \right\} \\
&\geq \inf_{s \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left(\sum_j [x_j, \alpha_j][s, \gamma] \right) \right\} \right\} \\
&= (\mu^+)^{+} \left(\sum_j [x_j, \alpha_j] \right).
\end{aligned}$$

Thus, $(\mu^+)^{+} \subseteq \mu$ and so $(\mu^+)^{+} = \mu$. Therefore, the correspondence $\sigma \mapsto \sigma^{+}$ is a bijection.

Now, let $\sigma_1, \sigma_2 \in \text{Fh-I}(S)$ be such that $\sigma_1 \subseteq \sigma_2$. Then

$$\sigma_1^{+} \left(\sum_i [x_i, \alpha_i] \right) = \inf_{s \in S} \left\{ \sigma_1 \left(\sum_i x_i \alpha_i s \right) \right\} \leq \inf_{s \in S} \left\{ \sigma_2 \left(\sum_i x_i \alpha_i s \right) \right\} = \sigma_2^{+} \left(\sum_i [x_i, \alpha_i] \right),$$

for all $\sum_i [x_i, \alpha_i] \in L$. Thus, $\sigma_1^{+} \subseteq \sigma_2^{+}$. Similarly, we can deduce that if

$\mu_1 \subseteq \mu_2$, where $\mu_1, \mu_2 \in \text{Fh-I}(L)$. Then $\mu_1^{+} \subseteq \mu_2^{+}$. We show that $(\sigma_1 \oplus \sigma_2)^{+} = \sigma_1^{+} \oplus \sigma_2^{+}$ and $(\sigma_1 \cap \sigma_2)^{+} = \sigma_1^{+} \cap \sigma_2^{+}$.

Let $\sum_i [a_i, \alpha_i] \in L$. Then

$$\begin{aligned}
& ((\sigma_1 \oplus \sigma_2)^{+'}) \left(\sum_i [a_i, \alpha_i] \right) \\
&= \inf_{s \in S} \left\{ (\sigma_1 \oplus \sigma_2) \left(\sum_i a_i \alpha_i s \right) \right\} \\
&= \inf_{s \in S} \left\{ \sup \left\{ \min \left\{ \sigma_1 \left(\sum_k x_k \delta_k s \right), \sigma_2 \left(\sum_j y_j \beta_j s \right) \right\} : \sum_i a_i \alpha_i s = \sum_k x_k \delta_k s + \sum_j y_j \beta_j s \right\} \right\} \\
&= \sup \left\{ \min \left\{ \inf_{s \in S} \left\{ \sigma_1 \left(\sum_k x_k \delta_k s \right) \right\}, \inf_{s \in S} \left\{ \sigma_2 \left(\sum_j y_j \beta_j s \right) \right\} \right\} \right\} \\
&= \sup \left\{ \min \left\{ \sigma_1^{+'} \left(\sum_k [x_k, \delta_k] \right), \sigma_2^{+'} \left(\sum_j [y_j, \beta_j] \right) \right\} \right\} \\
&= (\sigma_1^{+'} \oplus \sigma_2^{+'}) \left(\sum_i [a_i, \alpha_i] \right).
\end{aligned}$$

Thus $(\sigma_1 \oplus \sigma_2)^{+'} = \sigma_1^{+'} \oplus \sigma_2^{+'}$. Again, we have

$$\begin{aligned}
(\sigma_1 \cap \sigma_2)^{+'} \left(\sum_i [a_i, \alpha_i] \right) &= \inf_{s \in S} \left\{ (\sigma_1 \cap \sigma_2) \left(\sum_i a_i \alpha_i s \right) \right\} \\
&= \inf_{s \in S} \left\{ \min \left\{ \sigma_1 \left(\sum_i a_i \alpha_i s \right), \sigma_2 \left(\sum_i a_i \alpha_i s \right) \right\} \right\} \\
&= \min \left\{ \inf_{s \in S} \left\{ \sigma_1 \left(\sum_i a_i \alpha_i s \right) \right\}, \inf_{s \in S} \left\{ \sigma_2 \left(\sum_i a_i \alpha_i s \right) \right\} \right\} \\
&= \min \left\{ \sigma_1^{+'} \left(\sum_i [a_i, \alpha_i] \right), \sigma_2^{+'} \left(\sum_i [a_i, \alpha_i] \right) \right\} \\
&= (\sigma_1^{+'} \cap \sigma_2^{+'}) \left(\sum_i [a_i, \alpha_i] \right).
\end{aligned}$$

So $(\sigma_1 \cap \sigma_2)^{+'} = \sigma_1^{+'} \cap \sigma_2^{+'}$. Therefore, the mapping $\sigma \mapsto \sigma^{+'}$ is a lattice isomorphism. \square

Similarly, we can obtain the following theorem.

Theorem 3.9. *The lattice of all fuzzy h -ideals of S and the lattice of all fuzzy h -ideals of R are isomorphic via the mapping $\sigma \mapsto \sigma^{*'}$, where $\sigma \in Fh-I(S)$ and $\sigma^{*'} \in Fh-I(R)$.*

Corollary 3.10. $FLh-I(L)$, $FRh-I(L)$, $FLh-I(R)$ and $FRh-I(R)$ are complete lattices.

Proof. The corollary follows from the above theorems and the fact that FLh-I(S), FRh-I(S), Fh-I(S) are complete lattices [12]. \square

Now, by routine verification the following lemmas can be obtained.

Lemma 3.11. Let I be an h -ideal (left h -ideal, right h -ideal) of a Γ -hemiring S and λ_I be the characteristic function of I . Then $(\lambda_I)^{+'} = \lambda_{(I^{+'})}$.

Lemma 3.12. Let I be an h -ideal (left h -ideal, right h -ideal) of the left operator hemiring L of a Γ -hemiring S and λ_I be the characteristic function of I . Then $(\lambda_I)^+ = \lambda_{(I^+)}$.

Lemma 3.13. Let I be an h -ideal (left h -ideal, right h -ideal) of a Γ -hemiring S and λ_I be the characteristic function of I . Then $(\lambda_I)^{*'} = \lambda_{(I^{*'})}$.

Lemma 3.14. Let I be an h -ideal (left h -ideal, right h -ideal) of the right operator hemiring R of a Γ -hemiring S and λ_I be the characteristic function of I . Then $(\lambda_I)^* = \lambda_{(I^*)}$.

Now, we revisit the following theorem which is due to Dutta and Sardar [2].

Theorem 3.15. *The lattice of all h -ideals of S and the lattices of all h -ideals of L are isomorphic via the mapping $I \mapsto I^{+'}$, where I denotes an h -ideal of S .*

Proof. First, we show that the mapping $I \mapsto I^{+'}$ is one-one. Let I_1 and I_2 be two h -ideals of S such that $I_1 \neq I_2$. Then λ_{I_1} and λ_{I_2} are fuzzy h -ideals of S , where λ_{I_1} and λ_{I_2} are characteristic functions of I_1 and I_2 , respectively. Evidently, $\lambda_{I_1} \neq \lambda_{I_2}$. Then by Theorem 3.8, $\lambda_{I_1}^{+'} \neq \lambda_{I_2}^{+'}$. Hence by Lemma 3.11, $\lambda_{I_1^{+'}} \neq \lambda_{I_2^{+'}}$ whence $I_1^{+'} \neq I_2^{+'}$. Consequently, the mapping $I \mapsto I^{+'}$ is one-one. Now, let J be an h -ideal of L . Then λ_J is a fuzzy h -ideal of L . By Proposition 3.4 and Theorem 3.8, λ_J^+ is a fuzzy h -ideal of S . Now, by Lemma 3.12, $\lambda_J^+ = \lambda_{J^+}$ and consequently, J^+ is an h -ideal of S . Thus, the mapping is onto.

Now, let I_1, I_2 be two h -ideals of S such that $I_1 \subseteq I_2$. Then $\lambda_{I_1} \subseteq \lambda_{I_2}$ and by Theorem 3.8, $\lambda_{I_1}^{+'} \subseteq \lambda_{I_2}^{+'}$ and by Lemma 3.11, $\lambda_{I_1^{+'}} \subseteq \lambda_{I_2^{+'}}$ and consequently $I_1^{+'} \subseteq I_2^{+'}$. Thus, the mapping is inclusion preserving. Hence the theorem. \square

Similarly, we can revisit the following theorem:

Theorem 3.16. *The lattice of all h -ideals of S and the lattice of all h -ideals of R are isomorphic via the mapping $I \mapsto I^{*'}$, where I is an h -ideal of S .*

Proposition 3.17. For any two fuzzy h -ideals μ and ν of S , $(\mu \circ_h \nu)^{+'} = ((\mu)^{+'} \circ_h (\nu)^{+'})$.

Proof. Suppose that

$$\sum_i [x_i, e_i], \left(\sum_i [a_i, \alpha_i] \right)_j, \sum_i [z_i, \eta_i], \left(\sum_i [b_i, \beta_i] \right)_j, \left(\sum_i [c_i, \gamma_i] \right)_j, \left(\sum_i [d_i, \delta_i] \right)_j \in L$$

be such that

$$\begin{aligned} & \sum_i [x_i, e_i] + \sum_j \left(\sum_i [a_i, \alpha_i] \right)_j \left(\sum_i [c_i, \gamma_i] \right)_j + \sum_i [z_i, \eta_i] \\ &= \sum_j \left(\sum_i [b_i, \beta_i] \right)_j \left(\sum_i [d_i, \delta_i] \right)_j + \sum_i [z_i, \eta_i]. \end{aligned}$$

Then

$$\begin{aligned} & ((\mu)^+{}' o_h(\nu)^+{}') \left(\sum_i [x_i, e_i] \right) \\ &= \sup \left\{ \min_j \left\{ (\mu)^+{}' \left(\left(\sum_i [a_i, \alpha_i] \right)_j \right), (\nu)^+{}' \left(\left(\sum_i [c_i, \gamma_i] \right)_j \right), \right. \right. \\ & \quad \left. \left. (\mu)^+{}' \left(\left(\sum_i [b_i, \beta_i] \right)_j \right), (\nu)^+{}' \left(\left(\sum_i [d_i, \delta_i] \right)_j \right) \right\} \right\} \\ &= \sup \left\{ \min_j \left\{ \inf_{s \in S} \left\{ \mu \left(\left(\sum_i a_i \alpha_i s \right)_j \right) \right\}, \inf_{s \in S} \left\{ \nu \left(\left(\sum_i c_i \gamma_i s \right)_j \right) \right\}, \right. \right. \\ & \quad \left. \left. \inf_{s \in S} \left\{ \mu \left(\left(\sum_i b_i \beta_i s \right)_j \right) \right\}, \inf_{s \in S} \left\{ \nu \left(\left(\sum_i d_i \delta_i s \right)_j \right) \right\} \right\} \right\} \\ &= \inf_{s \in S} \left\{ \sup \left\{ \min_j \left\{ \mu \left(\left(\sum_i a_i \alpha_i s \right)_j \right), \nu \left(\left(\sum_i c_i \gamma_i s \right)_j \right), \right. \right. \right. \right. \\ & \quad \left. \left. \mu \left(\left(\sum_i b_i \beta_i s \right)_j \right), \nu \left(\left(\sum_i d_i \delta_i s \right)_j \right) \right\} \right\} \\ &= \inf_{s \in S} \left\{ (\mu o_h \nu) \left(\sum_i x_i e_i s \right) \right\} \\ &= (\mu o_h \nu)^+{}' \left(\sum_i [x_i, e_i] \right). \end{aligned}$$

□

Remark 3.18. Similarly, we can show that for any two fuzzy h -ideals μ and ν of S , $(\mu \Gamma_h \nu)^+{}' = \mu^+{}' \Gamma_h \nu^+{}'$

Proof. The proof is similar to Proposition 3.17 \square

Proposition 3.19. If ζ is a prime (respectively, semiprime) fuzzy h -ideal of S , then $\zeta^{+'}$ (respectively, ζ^{*}) is a prime (respectively, semiprime) fuzzy h -ideal of L (respectively, R).

Proof. Suppose that ζ is a prime fuzzy h -ideal of S and $\mu^{+'}, \nu^{+'}$ be fuzzy h -ideals of L such that $\mu^{+'}\Gamma_h\nu^{+'} \subseteq \zeta^{+'}$. Then by using the above remark we obtain $(\mu\Gamma_h\nu)^{+'} \subseteq \zeta^{+'}$ which implies that $(\mu\Gamma_h\nu) \subseteq \zeta$. Since ζ is a prime fuzzy h -ideal of S , then $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$, whence $\mu^{+'} \subseteq \zeta^{+'}$ or $\nu^{+'} \subseteq \zeta^{+'}$. Therefore, $\zeta^{+'}$ is a prime fuzzy h -ideal of L . Similarly, we can prove the result for R . Now, for semiprime fuzzy h -ideal the proof follows in a similar way. \square

Proposition 3.20. If ζ is a prime (respectively, semiprime) fuzzy h -ideal of L (respectively, R), then $\zeta^+(resp. \zeta^*)$ is a prime (respectively, semiprime) fuzzy h -ideal of S .

Proof. The proof follows by routine verification. \square

Proposition 3.21. If μ is a fuzzy h -bi-ideal of S , then $\mu^{+'}$ (respectively, μ^{*}) is a fuzzy h -bi-ideal of L (respectively, R).

Proof. Suppose that μ is a fuzzy h -bi-ideal of S and $\sum_i [x_i, \alpha_i], \sum_i [y_i, \beta_i], \sum_i [z_i, \gamma_i] \in L$.

Then by Proposition 3.5, we obtain

$$\mu^{+'} \left(\left(\sum_i [x_i, \alpha_i] \right) + \left(\sum_i [y_i, \beta_i] \right) \right) \geq \min \left\{ \mu^{+'} \left(\sum_i [x_i, \alpha_i] \right), \mu^{+'} \left(\sum_i [y_i, \beta_i] \right) \right\}$$

and

$$\mu^{+'} \left(\left(\sum_i [x_i, \alpha_i] \right) \left(\sum_i [y_i, \beta_i] \right) \right) \geq \min \left\{ \mu^{+'} \left(\sum_i [x_i, \alpha_i] \right), \mu^{+'} \left(\sum_i [y_i, \beta_i] \right) \right\}.$$

Now, suppose that $\sum_i [x_i, e_i], \sum_i [a_i, \alpha_i], \sum_i [z_i, \delta_i], \sum_i [b_i, \beta_i] \in L$ are such that

$$\sum_i [x_i, e_i] + \sum_i [a_i, \alpha_i] + \sum_i [z_i, \delta_i] = \sum_i [b_i, \beta_i] + \sum_i [z_i, \delta_i].$$

Since $\mu^{+'}$ is a fuzzy h -ideal of L , we have

$$\mu^{+'} \left(\sum_i [x_i, e_i] \right) \geq \min \left\{ \mu^{+'} \left(\sum_i [a_i, \alpha_i] \right), \mu^{+'} \left(\sum_i [b_i, \beta_i] \right) \right\}.$$

Now, we obtain

$$\begin{aligned}\mu^{+'} \left(\left(\sum_i [x_i, \alpha_i] \right) \left(\sum_i [y_i, \beta_i] \right) \left(\sum_i [z_i, \gamma_i] \right) \right) &= \mu^{+'} \left(\sum_i [x_i, \alpha_i] \sum_i [y_i \beta_i z_i, \gamma_i] \right) \\ &\geq \mu^{+'} \left(\sum_i [x_i, \alpha_i] \right).\end{aligned}$$

Similarly,

$$\begin{aligned}\mu^{+'} \left(\left(\sum_i [x_i, \alpha_i] \right) \left(\sum_i [y_i, \beta_i] \right) \left(\sum_i [z_i, \gamma_i] \right) \right) &= \mu^{+'} \left(\sum_i [x_i, \alpha_i y_i \beta_i] \sum_i [z_i, \gamma_i] \right) \\ &\geq \mu^{+'} \left(\sum_i [z_i, \gamma_i] \right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\mu^{+'} \left(\left(\sum_i [x_i, \alpha_i] \right) \left(\sum_i [y_i, \beta_i] \right) \left(\sum_i [z_i, \gamma_i] \right) \right) \\ &\geq \min \left\{ \mu^{+'} \left(\sum_i [x_i, \alpha_i] \right), \mu^{+'} \left(\sum_i [z_i, \gamma_i] \right) \right\}.\end{aligned}$$

Hence, $\mu^{+'}$ is a fuzzy h -bi-ideal of L . Similarly, we can prove the result for R . \square

Proposition 3.22. If μ is a fuzzy h -bi-ideal of L (respectively, R), then μ^+ (respectively, μ^*) is also a fuzzy h -bi-ideal of S .

Proposition 3.23. If μ is a fuzzy h -quasi-ideal of S , then $\mu^{+'}$ (respectively, $\mu^{*'}$) is a fuzzy h -quasi-ideal of L (respectively, R).

Proof. Suppose that μ is a fuzzy h -quasi-ideal of S and $\sum_i [x_i, \alpha_i], \sum_i [y_i, \beta_i], \sum_i [z_i, \gamma_i] \in L$. By Proposition 3.5, we obtain

$$\mu^{+'} \left(\sum_i [x_i, \alpha_i] + \sum_i [y_i, \beta_i] \right) \geq \min \left\{ \mu^{+'} \left(\sum_i [x_i, \alpha_i] \right), \mu^{+'} \left(\sum_i [y_i, \beta_i] \right) \right\}.$$

If

$$\sum_i [x_i, e_i] + \sum_i [a_i, \alpha_i] + \sum_i [z_i, \delta_i] = \sum_i [b_i, \beta_i] + \sum_i [z_i, \delta_i],$$

for $\sum_i [x_i, e_i], \sum_i [a_i, \alpha_i], \sum_i [z_i, \delta_i], \sum_i [b_i, \beta_i] \in L$, then

$$\mu^{+'} \left(\sum_i [x_i, e_i] \right) \geq \min \left\{ \mu^{+'} \left(\sum_i [a_i, \alpha_i] \right), \mu^{+'} \left(\sum_i [b_i, \beta_i] \right) \right\}.$$

Let χ_S be the characteristic function of S . Then by using Proposition 3.17 and Theorem 3.8 we deduce that

$$\left(\mu^{+'} o_h \chi_S^{+'}\right) \cap \left(\chi_S^{+'} o_h \mu^{+'}\right) = (\mu o_h \chi_S)^{+'} \cap (\chi_S o_h \mu)^{+'} = ((\mu o_h \chi_S) \cap (\chi_S o_h \mu))^{+'} \subseteq \mu^{+'}.$$

Thus, $\mu^{+'}$ is a fuzzy h -quasi-ideal of L . Similarly, we can prove the result for R . \square

Proposition 3.24. If μ is a fuzzy h -quasi-ideal of L (respectively, R), then μ^+ (respectively, μ^*) is also a fuzzy h -quasi-ideal of S .

4 Correspondence of Cartesian Product of Fuzzy h -ideals

Let $\{S_i\}_{i \in I}$ be a family of Γ -hemirings. We define addition (+) and multiplication (\cdot) on the cartesian product $\prod_{i \in I} S_i$ as follows :

$$(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I},$$

$$(x_i)_{i \in I} \alpha (y_i)_{i \in I} = (x_i \alpha y_i)_{i \in I},$$

for all $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} S_i$ and for all $\alpha \in \Gamma$. Then $\prod_{i \in I} S_i$ becomes a Γ -hemiring.

Definition 4.1. [1] Let μ and σ be two fuzzy subsets of a set X . Then the *cartesian product* of μ and σ is defined by

$$(\mu \times \sigma)(x, y) = \min\{\mu(x), \sigma(y)\},$$

for all $x, y \in X$.

Definition 4.2. Let $\mu \times \sigma$ be the cartesian product of two fuzzy subsets μ and σ of R . Then the corresponding cartesian product $(\mu \times \sigma)^*$ of $S \times S$ is defined by

$$(\mu \times \sigma)^*(x, y) = \inf_{\alpha, \beta \in \Gamma} \{(\mu \times \sigma)([\alpha, x], [\beta, y])\},$$

where $x, y \in S$.

Definition 4.3. Let $\mu \times \sigma$ be the cartesian product of two fuzzy subsets μ and σ of S . Then the corresponding cartesian product $(\mu \times \sigma)^{+'}$ of $R \times R$ is defined by

$$(\mu \times \sigma)^{+'} \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right) = \inf_{s_i, s_j \in S} \left\{ (\mu \times \sigma) \left(\sum_{i=1}^n s_i \alpha_i x_i, \sum_{j=1}^m s_j \beta_j y_j \right) \right\},$$

where $\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \in R$.

Definition 4.4. Let $\mu \times \sigma$ be the cartesian product of two fuzzy subsets μ and σ of L . Then the corresponding cartesian product $(\mu \times \sigma)^+$ of $S \times S$ is defined by

$$(\mu \times \sigma)^+(x, y) = \inf_{\alpha, \beta \in \Gamma} \{(\mu \times \sigma)([x, \alpha], [y, \beta])\},$$

where $x, y \in S$.

Definition 4.5. Let $\mu \times \sigma$ be the cartesian product of two fuzzy subsets μ and σ of S . Then the corresponding cartesian product $(\mu \times \sigma)^{+'}$ of $R \times R$ is defined by

$$(\mu \times \sigma)^{+'} \left(\sum_{i=1}^n [x_i, \alpha_i], \sum_{j=1}^m [y_j, \beta_j] \right) = \inf_{s_i, s_j \in S} \left\{ (\mu \times \sigma) \left(\sum_{i=1}^n x_i \alpha_i s_i, \sum_{j=1}^m y_j \beta_j s_j \right) \right\},$$

where $\sum_{i=1}^n [x_i, \alpha_i], \sum_{j=1}^m [y_j, \beta_j] \in L$.

Proposition 4.6. Let μ, μ', ν, ν' be four fuzzy h -ideals of S . Then

$$(\mu \times \mu') \Gamma_h (\nu \times \nu') = (\mu \Gamma_h \nu) \times (\mu' \Gamma_h \nu').$$

Proof. Let $(x, y) \in S \times S$ be such that

$$(x, y) + (a, c) \gamma (a', c') + (z, z') = (b, d) \delta (b', d') + (z, z'),$$

where $a, c, a', c', z, z', b, d, b', d' \in S$ and $\gamma, \delta \in \Gamma$. Then, $(x, y) + (a \gamma a', c \gamma c') + (z, z') = (b \delta b', d \delta d') + (z, z')$ which implies that $(x + a \gamma a' + z, y + c \gamma c' + z') = (b \delta b' + z, d \delta d' + z')$. Now, we have

$$\begin{aligned} & (\mu \times \mu') \Gamma_h (\nu \times \nu') (x, y) \\ &= \sup \left\{ \min \left\{ (\mu \times \mu')(a, c), (\mu \times \mu')(b, d), (\nu \times \nu')(a', c'), (\nu \times \nu')(b', d') \right\} \right\} \\ &= \sup \left\{ \min \left\{ \min \left\{ \mu(a), \mu'(c) \right\}, \min \left\{ \mu(b), \mu'(d) \right\}, \min \left\{ \nu(a'), \nu'(c') \right\}, \min \left\{ \nu(b'), \nu'(d') \right\} \right\} \right\} \\ &= \min \left\{ \sup_{x+a\gamma a'+z=b\delta b'+z} \left\{ \min \left\{ \mu(a), \mu(b), \nu(a'), \nu(b') \right\} \right\}, \right. \\ & \quad \left. \sup_{y+c\gamma c'+z'=d\delta d'+z'} \left\{ \min \left\{ \mu'(c), \mu'(d), \nu'(c'), \nu'(d') \right\} \right\} \right\} \\ &= \min \left\{ (\mu \Gamma_h \nu)(x), (\mu' \Gamma_h \nu')(y) \right\} \\ &= \left\{ (\mu \Gamma_h \nu) \times (\mu' \Gamma_h \nu') \right\} (x, y). \end{aligned}$$

Hence, the proof is completed. \square

Note that in this section, we prove the results for the Γ -hemiring S and its right operator hemiring R . Similar results hold for the Γ -hemiring S and its left operator hemiring L .

Proposition 4.7. Let μ and σ be two fuzzy subsets of R (respectively, of L)[the right (left) operator hemiring of the Γ -hemiring S]. Then $(\mu \times \sigma)^* = \mu^* \times \sigma^*$ (respectively, $(\mu \times \sigma)^+ = \mu^+ \times \sigma^+$).

Proof. Let $x, y \in S$. Then

$$\begin{aligned} (\mu \times \sigma)^*(x, y) &= \inf_{\alpha, \beta \in \Gamma} \{(\mu \times \sigma)([\alpha, x], [\beta, y])\} \\ &= \inf_{\alpha, \beta \in \Gamma} \{\min\{\mu([\alpha, x]), \sigma([\beta, y])\}\} \\ &= \min \left\{ \inf_{\alpha \in \Gamma} \{\mu([\alpha, x])\}, \inf_{\beta \in \Gamma} \{\sigma([\beta, y])\} \right\} \\ &= \min \{\mu^*(x), \sigma^*(y)\} \\ &= (\mu^* \times \sigma^*)(x, y). \end{aligned}$$

Consequently, $(\mu \times \sigma)^* = \mu^* \times \sigma^*$. Similarly, we can show that $(\mu \times \sigma)^+ = \mu^+ \times \sigma^+$. \square

Proposition 4.8. Let μ and σ be two fuzzy subsets of S . Then $(\mu \times \sigma)^{*'} = \mu^{*'} \times \sigma^{*'}$ and $(\mu \times \sigma)^{+'} = \mu^{+'} \times \sigma^{+'}$.

Proof. Let $\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \in R$. Then

$$\begin{aligned} (\mu \times \sigma)^{*'} \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right) &= \inf_{s_i, s_j \in S} \left\{ (\mu \times \sigma) \left(\sum_{i=1}^n s_i \alpha_i x_i, \sum_{j=1}^m s_j \beta_j y_j \right) \right\} \\ &= \inf_{s_i, s_j \in S} \left\{ \min \left\{ \mu \left(\sum_{i=1}^n s_i \alpha_i x_i \right), \left(\sum_{j=1}^m s_j \beta_j y_j \right) \right\} \right\} \\ &= \min \left\{ \inf_{s_i \in S} \left\{ \mu \left(\sum_{i=1}^n s_i \alpha_i x_i \right) \right\}, \inf_{s_j \in S} \left\{ \sigma \left(\sum_{j=1}^m s_j \beta_j y_j \right) \right\} \right\} \\ &= \min \left\{ \mu^{*'} \left(\sum_{i=1}^n [\alpha_i, x_i] \right), \sigma^{*'} \left(\sum_{j=1}^m [\beta_j, y_j] \right) \right\} \\ &= (\mu^{*'} \times \sigma^{*'}) \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right). \end{aligned}$$

Consequently, $(\mu \times \sigma)^{*'} = \mu^{*'} \times \sigma^{*'}$. Similarly, we can show that $(\mu \times \sigma)^{+'} = \mu^{+'} \times \sigma^{+'}$. \square

Proposition 4.9. Let μ and σ be two fuzzy h -ideal of R (respectively, L). Then $\mu^* \times \sigma^*$ (respectively, $\mu^+ \times \sigma^+$) is a fuzzy h -ideal of $S \times S$.

Proof. Since μ, σ are fuzzy h -ideals of R , by Proposition 3.6, μ^*, σ^* are fuzzy h -ideals of S and by Theorem 35 of [12], we deduce that $\mu^* \times \sigma^*$ is a fuzzy h -ideal of $S \times S$. In a similar way, we can prove the result for L . \square

Proposition 4.10. Let μ and σ be two prime (semiprime) fuzzy h -ideals of R (respectively, of L). Then $\mu^* \times \sigma^*$ (respectively, $\mu^+ \times \sigma^+$) is prime (semiprime) fuzzy h -ideal of $S \times S$.

Proof. Suppose that μ and σ are two prime fuzzy h -ideals of R . By Proposition 4.9. we see that $\mu^* \times \sigma^*$ is a fuzzy h -ideal of $S \times S$. In order to show that $\mu^* \times \sigma^*$ is prime, suppose that $\theta, \theta', \eta, \eta' \in \text{Fh-I}(S)$ such that $(\theta \times \theta') \Gamma_h (\eta \times \eta') \subseteq \mu^* \times \sigma^*$. Then by Proposition 4.6, we obtain $(\theta \Gamma_h \eta) \times (\theta' \Gamma_h \eta') \subseteq \mu^* \times \sigma^*$. Therefore, $(\theta \Gamma_h \eta) \subseteq \mu^*$ and $(\theta' \Gamma_h \eta') \subseteq \sigma^*$. Hence $\theta \subseteq \mu^*$ or $\eta \subseteq \mu^*$ and $\theta' \subseteq \sigma^*$ or $\eta' \subseteq \sigma^*$, that is, $\theta \times \theta' \subseteq \mu^* \times \sigma^*$ or $\eta \times \eta' \subseteq \mu^* \times \sigma^*$. So, $\mu^* \times \sigma^*$ is a prime fuzzy h -ideal of $S \times S$. Similarly, we can prove the result for semiprime fuzzy h -ideal and the left operator hemiring L . \square

By suitable modification of above argument we obtain the following result.

Proposition 4.11. Let μ and σ be two fuzzy h -ideals (prime fuzzy h -ideals, semiprime fuzzy h -ideals) of S . Then $\mu^{*'} \times \sigma^{*'}$ (respectively, $\mu^{+'} \times \sigma^{+'}$) is a fuzzy h -ideal (respectively, prime fuzzy h -ideals, semiprime fuzzy h -ideals) of $R \times R$ (respectively, $L \times L$).

Theorem 4.12. Let S be a Γ -hemiring with unities and R be its right operator hemiring. Then there exists an inclusion preserving bijection $\mu \times \sigma \longmapsto \mu^{*'} \times \sigma^{*'}$ between the set of all cartesian product of fuzzy h -ideals (respectively, prime fuzzy h -ideals, semiprime fuzzy h -ideals) of S and the set of all cartesian product of fuzzy h -ideals (respectively, prime fuzzy h -ideals, semiprime fuzzy h -ideals) of R , where μ and σ are fuzzy h -ideals (respectively, prime fuzzy h -ideals, semiprime fuzzy h -ideals) of S .

Proof. Suppose that μ and σ are fuzzy h -ideals of S and $x, y \in S$. Then

$$\begin{aligned}
(\mu^{*'} \times \sigma^{*'})^*(x, y) &= \inf_{\alpha, \beta \in \Gamma} \left\{ (\mu^{*'} \times \sigma^{*'})([\alpha, x], [\beta, y]) \right\} \\
&= \inf_{\alpha, \beta \in \Gamma} \left\{ \min \left\{ \mu^{*'}([\alpha, x]), \sigma^{*'}([\beta, y]) \right\} \right\} \\
&= \min \left\{ \inf_{\alpha \in \Gamma} \left\{ \mu^{*'}([\alpha, x]) \right\}, \inf_{\beta \in \Gamma} \left\{ \sigma^{*'}([\beta, y]) \right\} \right\} \\
&= \min \left\{ \inf_{\alpha \in \Gamma} \left\{ \inf_{s_1 \in S} \left\{ \mu(s_1 \alpha x) \right\} \right\}, \inf_{\beta \in \Gamma} \left\{ \inf_{s_2 \in S} \left\{ \sigma(s_2 \beta y) \right\} \right\} \right\} \\
&\geq \min \{ \mu(x), \sigma(y) \} \\
&= (\mu \times \sigma)(x, y).
\end{aligned}$$

Therefore, $\mu \times \sigma \subseteq (\mu^{*'} \times \sigma^{*'})^*$. Let $[e, \delta]$ be the strong left unity of S . Then $e\delta x = x$ and $e\delta y = y$, for all $x, y \in S$. Now, we have

$$\begin{aligned}
(\mu \times \sigma)(x, y) &= \min\{\mu(x), \sigma(y)\} = \min\{\mu(e\delta x), \sigma(e\delta y)\} \\
&\geq \min \left\{ \inf_{\alpha \in \Gamma} \left\{ \inf_{s_1 \in S} \{\mu(s_1 \alpha x)\} \right\}, \inf_{\beta \in \Gamma} \left\{ \inf_{s_2 \in S} \{\sigma(s_2 \beta y)\} \right\} \right\} \\
&= \min \left\{ \inf_{\alpha \in \Gamma} \left\{ \mu^{*'}([\alpha, x]) \right\}, \inf_{\beta \in \Gamma} \left\{ \sigma^{*'}([\beta, y]) \right\} \right\} \\
&= \min \left\{ (\mu^{*'})^*(x), (\sigma^{*'})^*(y) \right\} \\
&= ((\mu^{*'})^* \times (\sigma^{*'})^*)(x, y) \\
&= (\mu^{*'} \times \sigma^{*'})^*(x, y).
\end{aligned}$$

So, $\mu \times \sigma \supseteq (\mu^{*'} \times \sigma^{*'})^*$. Hence, $\mu \times \sigma = (\mu^{*'} \times \sigma^{*'})^*$. Now, let μ and σ be two fuzzy h -ideals of R . Then

$$\begin{aligned}
&(\mu^{*'} \times \sigma^{*'})^{*'} \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right) \\
&= \inf_{s_i, s_j \in S} \left\{ (\mu^{*'} \times \sigma^{*'}) \left(\sum_{i=1}^n s_i \alpha_i x_i, \sum_{j=1}^m s_j \beta_j y_j \right) \right\} \\
&= \inf_{s_i, s_j \in S} \left\{ \min \left\{ \mu^{*'} \left(\sum_{i=1}^n s_i \alpha_i x_i \right), \sigma^{*'} \left(\sum_{j=1}^m s_j \beta_j y_j \right) \right\} \right\} \\
&= \min \left\{ \inf_{s_i \in S} \left\{ \mu^{*'} \left(\sum_{i=1}^n s_i \alpha_i x_i \right) \right\}, \inf_{s_j \in S} \left\{ \sigma^{*'} \left(\sum_{j=1}^m s_j \beta_j y_j \right) \right\} \right\} \\
&= \min \left\{ \inf_{s_i \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left([\gamma, \sum_{i=1}^n s_i \alpha_i x_i] \right) \right\} \right\}, \inf_{s_j \in S} \left\{ \inf_{\delta \in \Gamma} \left\{ \sigma \left([\delta, \sum_{j=1}^m s_j \beta_j y_j] \right) \right\} \right\} \right\} \\
&= \min \left\{ \inf_{s_i \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left(\sum_{i=1}^n [\gamma, s_i] \sum_{i=1}^n [\alpha_i, x_i] \right) \right\} \right\}, \inf_{s_j \in S} \left\{ \inf_{\delta \in \Gamma} \left\{ \sigma \left(\sum_{j=1}^m [\delta, s_j] \sum_{j=1}^m [\beta_j, y_j] \right) \right\} \right\} \right\} \\
&\geq \min \left\{ \mu \left(\sum_{i=1}^n [\alpha_i, x_i] \right), \sigma \left(\sum_{j=1}^m [\beta_j, y_j] \right) \right\} \\
&= (\mu \times \sigma) \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right).
\end{aligned}$$

Thus, we obtain $(\mu^{*'} \times \sigma^{*'})^{*'} \supseteq \mu \times \sigma$. Let $\sum_{k=1}^p [\gamma_k, f_k]$ be the right unity of S .

Then

$$\begin{aligned}
& (\mu \times \sigma) \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right) \\
&= \min \left\{ \mu \left(\sum_{i=1}^n [\alpha_i, x_i] \right), \sigma \left(\sum_{j=1}^m [\beta_j, y_j] \right) \right\} \\
&= \min \left\{ \mu \left(\sum_{i=1}^n [\alpha_i, x_i] \sum_{k=1}^p [\gamma_k, f_k] \right), \sigma \left(\sum_{j=1}^m [\beta_j, y_j] \sum_{k=1}^p [\gamma_k, f_k] \right) \right\} \\
&\geq \min \left\{ \inf_{s_i \in S} \left\{ \inf_{\gamma \in \Gamma} \left\{ \mu \left(\sum_{i=1}^n [\alpha_i, x_i] \sum_{i=1}^n [\gamma, s_i] \right) \right\} \right\}, \inf_{s_j \in S} \left\{ \inf_{\delta \in \Gamma} \left\{ \sigma \left(\sum_{j=1}^m [\beta_j, y_j] \sum_{j=1}^m [\delta, s_j] \right) \right\} \right\} \right\} \\
&\geq \min \left\{ (\mu^*)^* \left(\sum_{i=1}^n [\alpha_i, x_i] \right), (\sigma^*)^* \left(\sum_{j=1}^m [\beta_j, y_j] \right) \right\} \\
&= \left((\mu^*)^* \times (\sigma^*)^* \right) \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right) \\
&= (\mu^* \times \sigma^*)^* \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right).
\end{aligned}$$

Hence, $(\mu^* \times \sigma^*)^* \subseteq \mu \times \sigma$. Consequently, $(\mu^* \times \sigma^*)^* = \mu \times \sigma$. Thus, we see that the correspondence $\mu \times \sigma \mapsto \mu^* \times \sigma^*$ is a bijection. Now, suppose that $\mu_1, \mu_2, \sigma_1, \sigma_2$ are fuzzy h -ideals of S such that $\mu_1 \times \sigma_1 \subseteq \mu_2 \times \sigma_2$. Then

$$\begin{aligned}
& (\mu_1^* \times \sigma_1^*) \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right) \\
&= \inf_{s_i, s_j \in S} \left\{ (\mu_1 \times \sigma_1) \left(\sum_{i=1}^n s_i \alpha_i x_i, \sum_{j=1}^m s_j \beta_j y_j \right) \right\} \\
&\leq \inf_{s_i, s_j \in S} \left\{ (\mu_2 \times \sigma_2) \left(\sum_{i=1}^n s_i \alpha_i x_i, \sum_{j=1}^m s_j \beta_j y_j \right) \right\} \\
&= \inf_{s_i, s_j \in S} \left\{ \min \left\{ \mu_2 \left(\sum_{i=1}^n s_i \alpha_i x_i \right), \sigma_2 \left(\sum_{j=1}^m s_j \beta_j y_j \right) \right\} \right\} \\
&= \min \left\{ \inf_{s_i \in S} \left\{ \mu_2 \left(\sum_{i=1}^n s_i \alpha_i x_i \right) \right\}, \inf_{s_j \in S} \left\{ \sigma_2 \left(\sum_{j=1}^m s_j \beta_j y_j \right) \right\} \right\} \\
&= \min \left\{ \mu_2^* \left(\sum_{i=1}^n [\alpha_i, x_i] \right), \sigma_2^* \left(\sum_{j=1}^m [\beta_j, y_j] \right) \right\} \\
&= (\mu_2^* \times \sigma_2^*) \left(\sum_{i=1}^n [\alpha_i, x_i], \sum_{j=1}^m [\beta_j, y_j] \right).
\end{aligned}$$

Hence, $\mu_1^* \times \sigma_1^* \subseteq \mu_2^* \times \sigma_2^*$. Therefore, $\mu \times \sigma \mapsto \mu^* \times \sigma^*$ is an inclusion preserving bijection. Similarly, we can prove the results for prime fuzzy h -ideals and semiprime fuzzy h -ideals. \square

Conclusion: As a continuation of this paper we will study the correspondence of h -hemiregularity, h -intra-hemiregularity etc in operator hemirings.

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